

An exact universal amplitude ratio for percolation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2001 J. Phys. A: Math. Gen. 34 L759

(<http://iopscience.iop.org/0305-4470/34/50/105>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.101

The article was downloaded on 02/06/2010 at 09:49

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

An exact universal amplitude ratio for percolation**Katherine A Seaton**¹C N Yang Institute for Theoretical Physics, State University of New York, Stony Brook,
NY 11794-3840, USA

E-mail: k.seaton@latrobe.edu.au

Received 31 October 2001, in final form 16 November 2001

Published 7 December 2001

Online at stacks.iop.org/JPhysA/34/L759**Abstract**

The universal amplitude ratio \tilde{R}_ξ^+ for percolation in two dimensions is determined exactly using results for the dilute A model in regime 1, by way of a relationship with the q -state Potts model, with $q \leq 4$.

PACS numbers: 64.60.Ak, 05.50.+q

Percolation is a subject which can be described simply and yet has many important physical applications (see [1] and references therein). In statistical mechanics, one natural arena in which to study its inherent critical phenomena, percolation is related by way of the random cluster model [2] to the q -state Potts model with $q = 1$.

The *critical exponents* of the two-dimensional Potts model for $q \leq 4$, determined from numerical and renormalization group studies [3, 4], led to the identification [5] of the Potts model (in the scaling limit) for $q = 2, 3, 4$ with the $\phi_{2,1}$ perturbation of the minimal unitary series $\mathcal{M}(t, t + 1)$ by way of

$$\sqrt{q} = 2 \sin \left(\frac{\pi(t-1)}{2(t+1)} \right). \quad (1)$$

With care, one can extend the identification to $q = 1$.

On the other hand, for the associated *critical amplitudes* (except in the Ising case $q = 2$ where exact results are known [6]) the series and Monte Carlo results can vary widely [7], making it difficult to obtain reliable estimates for the universal amplitude ratios [8]. Recently numerical values for various amplitude ratios of the Potts model (including percolation) have been determined [9] by a technique of perturbed conformal field theory, specifically by the two-kink approximation in the form factor approach to the S -matrix of [10].

The only one of the amplitude ratios determined in [9] which will be considered in this Letter is the percolation amplitude [8, 9]

$$\tilde{R}_\xi^+ = \lim_{q \rightarrow 1} \left[\frac{\alpha(2-\alpha)(1-\alpha)\mathcal{A}_f}{(q-1)} \right]^{\frac{1}{2}} \xi_0^+. \quad (2)$$

¹ Permanent address: School of Mathematical and Statistical Sciences, La Trobe University, Victoria 3086, Australia.

Here ξ_0^+ is the leading term amplitude of the correlation length (above the critical temperature)

$$\xi \simeq \xi_0^+ \tau^{-\nu}$$

and \mathcal{A}_f and α come from the singular part of the free energy,

$$-f_s \simeq \mathcal{A}_f \tau^{2-\alpha}.$$

The percolation analogue of the free energy is the mean number of clusters, and of ξ , the pair connectivity. That \tilde{R}_ξ is universal, i.e. independent of metric factors associated with the reduced temperature $\tau \propto T - T_c$, follows from the scaling relation $2 - \alpha = d\nu$ in $d = 2$ dimensions.

There is an exactly solvable model, the dilute A_L model [13], which has also been identified with the $\phi_{2,1}$ perturbation of the minimal unitary series [14]. It is an L -state, interaction-round-a-face model, solved in four regimes, two of which provide off-critical extensions of the unitary minimal conformal field theories [14]. While regime 2 of the model is associated with the perturbation $\phi_{1,2}$, regime 1 of the model realizes (in the scaling limit) the perturbation $\phi_{2,1}$ of the minimal unitary series $\mathcal{M}(L+1, L+2)$.

That they are both identified with the same perturbation $\phi_{2,1}$ suggests a relationship at least between certain members of the dilute A_L hierarchy and the Potts models. In general the Potts model and the dilute A model represent different universality classes associated with $\phi_{2,1}$, just as in the field theory context there may be more than one S -matrix for a given perturbation. One simple way to see that the models have different internal symmetries is to count ground states: for the Potts model there are q of them, and for the dilute A model the number grows linearly with L , but the connection between L and q implied by setting $t = L+1$ in (1) is certainly not linear.

Nevertheless, on the basis of this relationship between models, weaker than shared universality class, it is still possible to determine one universal amplitude ratio for the Potts model, including percolation, from the dilute A model. In the language of perturbed conformal field theory, the free energy and the correlation length are related directly to the coupling constant g of the perturbation and the associated conformal weight Δ :

$$f_s \sim g^{\frac{d}{d-2\Delta}} \quad \xi \sim g^{\frac{-1}{d-2\Delta}}.$$

Thus when attention is confined to the correlation length amplitude ratio (2), the required quantities \mathcal{A}_f , ξ_0 and α (or equivalently Δ) relate solely to the perturbing operator. This operator is $\phi_{2,1}$ for both dilute A in regime 1 and the Potts model, and any universal observable associated only with it should be common [15].

From (1), percolation corresponds to $t = 2$, so the member of the dilute A hierarchy to be considered is $L = 1$. Since L labels the number of states in the dilute A_L model, it is properly an integer $L \geq 2$. However, it has long been realized that q can be treated as a continuous variable in an appropriate formulation of the Potts model [2] and it is in this spirit that we now take $L \rightarrow 1$ in the free energy and correlation length expressions of the dilute A model.

The leading term of the correlation length [16, 17] of the dilute A_L model in regime 1 is

$$\xi^{-1} \simeq 4\sqrt{3} p^{2(L+1)/3L}.$$

There is good reason to believe that this applies to the high-temperature regime for all L , although it was actually determined from L odd alone.

Although the singular part of the free energy of the dilute A model has been determined by both the inversion relation [14] and exact perturbative [16, 17] approaches, the coefficient has not previously been explicitly given; apart from $L = 2$, in regime 1 it may be written

$$-f_s \simeq \frac{4\sqrt{3} \sin(2\pi(L-1)/3L)}{\sin(\pi(L-2)/3L)} p^{4(L+1)/3L}. \quad (3)$$

This correctly gives the critical exponent $\alpha = -2/3$ for percolation by setting $L = 1$ in

$$\alpha = 2 - \frac{4(L+1)}{3L} = \frac{2(L-2)}{3L}.$$

The coefficient in (3) vanishes at $L = 1$, which agrees with the $q = 1$ Potts model amplitude given by Kaufman and Andelman [18].

The universal amplitude ratio \tilde{R}_ξ^+ , however, is finite and non-zero. Using trigonometric identities in (1) it is possible to write

$$q - 1 = 4 \sin\left(\frac{\pi(2t-1)}{3(t+1)}\right) \sin\left(\frac{\pi(t-2)}{3(t+1)}\right)$$

in which we now set $t = L + 1$. Using the last four expressions in (2), and taking the limit $L \rightarrow 1$, gives the algebraic expression for percolation:

$$\tilde{R}_\xi^+ = \left[\frac{(L-2)(L+1)(L+4)(L+2)}{27\sqrt{3}L^4 \sin\left(\frac{\pi(2L+1)}{3(L+2)}\right) \sin\left(\frac{\pi(L-2)}{3L}\right)} \right]_{L=1}^{\frac{1}{2}} = \left[\frac{40}{27\sqrt{3}} \right]^{\frac{1}{2}}. \quad (4)$$

To compare this result with the numerical field theory result of [9], it is to four figures

$$\tilde{R}_\xi^+ = 2^{\frac{3}{2}} 3^{-\frac{7}{4}} 5^{\frac{1}{2}} = 0.9248 \dots$$

Delfino and Cardy [9] quote a previous best estimate $\tilde{R}_\xi^+ \approx 1.1$ based on earlier series [11] and Monte Carlo [12] results. By the technique they adopt, they obtain for comparison the numerical value $\tilde{R}_\xi^+ = 0.926$. However, both of these values use the ‘second-moment’ definition of the correlation length, which differs in general from the ‘true’ correlation length used in this Letter; the ratio of the two correlation lengths is estimated in [9] to be ≈ 1.001 for $q = 1$. When this correction factor is included, to the accuracy stated the quoted numerical result of [9] agrees with the exact result (4). Numerical values for other universal amplitude ratios are also found in [9]; unlike (4) (which is in fact implicit in [9]), these do not all appear to be accessible exactly or from existing solvable models. Thus the value of knowing the reliability of the field theory approach, specifically the form factor approach at the two-kink level, is clear (see [19]).

A more detailed explanation of the relationship between the q -state Potts model and the dilute A model, together with the calculation of the universal correlation length amplitude ratio R_ξ^+ for the q -state Potts model for all integer $1 \leq q \leq 4$, will be given elsewhere [20].

This work was completed during sabbatical (OSP) leave spent at the Department of Mathematics and Statistics, University of Melbourne and at SUNY Stony Brook. The author particularly acknowledges Barry McCoy’s useful discussions and hospitality. She also thanks Aldo Delfino for informative comments on the first version of this Letter.

References

- [1] Stauffer D and Aharony A 1994 *Introduction to Percolation Theory* (London: Taylor and Francis)
- [2] Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536
- [3] den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
den Nijs M P M 1979 *Physica A* **95** 449
- [4] Nienhuis B, Reidel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L31
Nienhuis B, Reidel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L189
- [5] Dotsenko V I S and Fateev V A 1984 *Nucl. Phys. B* **240** 312
- [6] McCoy B M and Wu T T 1982 *The Two-Dimensional Ising Model* (Harvard) and references therein
- [7] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235

- [8] Privman V, Hohenberg P C and Aharony A 1991 *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J Lebowitz (New York: Academic) pp 4–134
- [9] Delfino G and Cardy J L 1998 *Nucl. Phys. B* **519** 551
- [10] Chim L and Zamolodchikov A B 1992 *Int. J. Mod. Phys. A* **7** 5317
- [11] Domb C and Pearce C J 1976 *J. Phys. A: Math. Gen.* **9** L137
- [12] Corsten M, Jan N and Jerrard R 1989 *Physica A* **156** 781
- [13] Warnaar S O, Nienhuis B and Seaton K A 1992 *Phys. Rev. Lett.* **69** 710
Warnaar S O, Nienhuis B and Seaton K A 1993 *Int. J. Mod. Phys. B* **7** 3727
- [14] Warnaar S O, Pearce P A, Seaton K A and Nienhuis B 1994 *J. Stat. Phys.* **74** 469
- [15] Delfino G 2000 *Nucl. Phys. B* **583** 597
- [16] Batchelor M T and Seaton K A 1997 *J. Phys. A: Math. Gen.* **30** L479
- [17] Batchelor M T and Seaton K A 1998 *Nucl. Phys. B* **520** 697
- [18] Kaufman M and Andelman D 1984 *Phys. Rev. B* **29** 4010
- [19] Cardy J L and Mussardo G 1993 *Nucl. Phys. B* **410** 451
- [20] Seaton K A 2001 A universal amplitude ratio for the $q \leq 4$ Potts model from a solvable lattice model *J. Stat. Phys.* submitted (hep-th/0111144)